Assignment 10

This homework is due *Thursday* Nov 12.

There are total 20 points in this assignment. 18 points is considered 100%. If you go over 18 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 5.3 and the end of 5.6 in Bartle–Sherbert.

(1) [2pt] (5.3.1) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in I$. Prove that there is a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.

(2) [2pt] (Part of 5.3.5) Show that the polynomial $x^4 + 7x^3 - 9$ has at least two real roots.

(3) [3pt] (5.3.6) Let f be continuous on the interval $[0, 1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. (*Hint:* Consider $g(x) = f(x) - f(x + \frac{1}{2})$.)
Note. Therefore, there are, at any time, antipodal points on the earth's equator that have the same temperature.

(4) [4pt] (5.3.13) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim\limits_{x\to-\infty} f = \lim\limits_{x\to+\infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or a minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained. (*Hint:* Pick M large enough and inspect how f behaves on an interval $[-M, M]$, on $\mathbb{R}\backslash[-M, M]$.)

(5) (a) [3pt] (5.3.11) Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W = \{x \in I : f(x) < 0\}$, and let $w = \sup W$. Prove that $f(w) = 0$. (This provides an alternate proof of Location of Roots Theorem.)

(b) [1pt] Why the same reasoning does not necessarily work if both $f(a) > 0, f(b) > 0$? (That is, find a precise place in the construction above that doesn't go through in such case.)

The next two problems are required for the introduction of rational power functions. In particular, in solving them you cannot use properties of rational powers (doing so would be a vicious circle), but rather only the definition of n-th root function (i.e., that for $x \geq 0$, $n \in \mathbb{N}$ we have $(x^{1/n})^n = (x^n)^{1/n} = x$).

(6) [2pt] ($\sim$Theorem 5.6.7) Let $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and let $x \in \mathbb{R}$, $x > 0$. Let $f(x) = x^m$, and $g(x)$ be the n-th root function, $g(x) = x^{1/n}$. Prove that $f(g(x)) = g(f(x))$, that is

$$(x^{1/n})^m = (x^m)^{1/n}.$$

(*Hint:* Denote $y = (x^{1/n})^m$. Show that $y^n = x^m$.)

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(7) (a) [1pt] Let $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $q \in \mathbb{N}$, and let $x \in \mathbb{R}$. Show that

$$(x^n)^{\frac{1}{nq}} = x^{\frac{1}{q}}.$$

(*Hint:* Denote LHS by y, show that $y^q = x$ using Problem (6).)

(b) [2pt] (∼Def. 5.6.6) Let $m, p \in \mathbb{Z}$, $n, q \in \mathbb{N}$, and $x \in \mathbb{R}$, $x > 0$. Show that if $\frac{m}{n} = \frac{p}{q}$, then

$$(x^{1/n})^m = (x^{1/q})^p.$$

(*Hint:* Extract root nq from the equality $x^{mq} = x^{np}$ (why does the equality take place, by the way?). For that, use the Problems (6) and (7a).)

Comment. This problem explains that the function x^r given by $x^r = (x^{1/n})^m$ for $r = m/n \in \mathbb{Q}$ is well-defined.